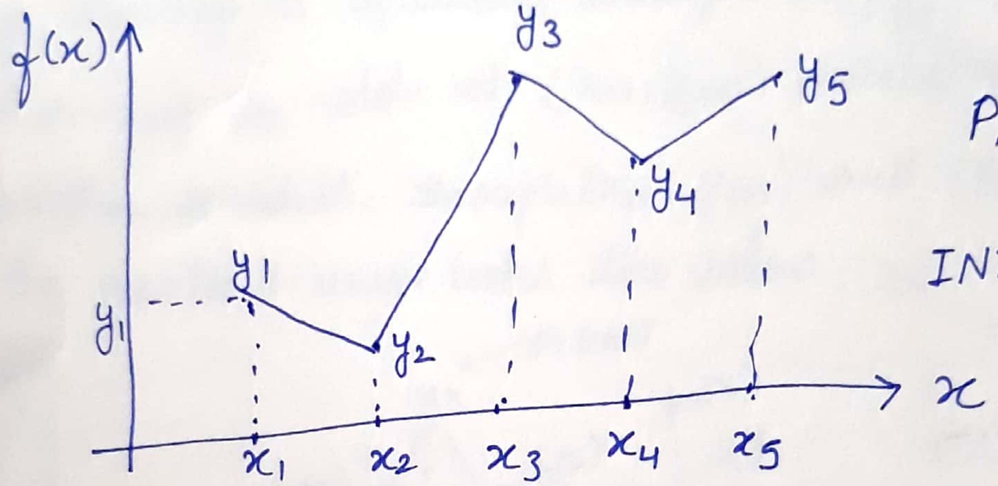
Spline Interpolation

Here we fit a graph to a set of points by using piecewise linear segments where the slope of the segment depends on the values of the function at the two closest points.



PIECEWISE LINEAR INTERPOLATION

In this case, the equation of a straight line connecting (x_i, y_i) and (x_{i+1}, y_{i+1}) is

$$f(x) = f(x_{i+1})\frac{(x-x_i)}{(x_{i+1}-x_i)} + f(x_i)\frac{(x-x_{i+1})}{(x_i-x_{i+1})}$$

Let $\frac{f(x_{i+1})}{(x_{i+1}-x_i)} = a_1$, $\frac{f(x_i)}{(x_i-x_{i+1})} = a_2$

then

$$f(x) = a_1(x-x_i) + a_2(x-x_{i+1})$$

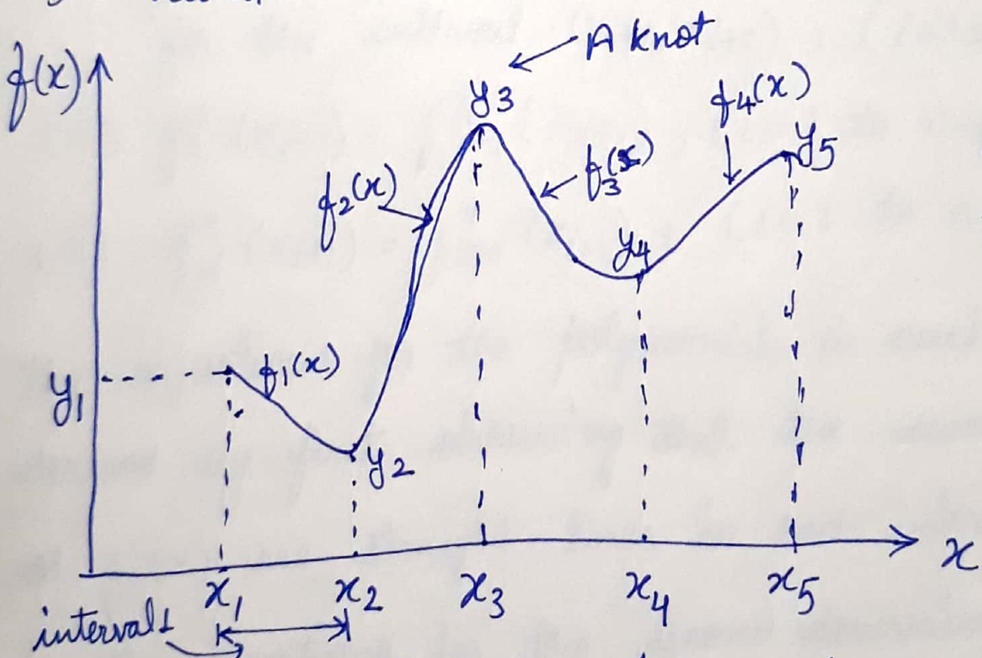
$$\Rightarrow f(x) = \frac{f(x_{i+1})(x-x_i) - f(x_i)(x_i-x_{i+1})}{(x_{i+1}-x_i)}$$

with $x_{i+1} \geq x \geq x_i$

CUBIC SPLINE

In the case of piecewise linear segments, the different line segments have different slopes and the resultant graph does not look smooth.

To overcome this problem, we solve this problem by drawing a quadratic through (x_i, y_i) and (x_{i+1}, y_{i+1}) such that its slope at (x_{i+1}, y_{i+1}) matches with that of another quadratic through (x_{i+1}, y_{i+1}) and (x_{i+2}, y_{i+2}).
The resultant curve looks like below:



CUBIC SPLINE INTERPOLATING FUNCTION

In this method the slope and curvature of the two cubics match at point (x_{i+1}, y_{i+1}). Such a graph is called a <u>Cubic Spline</u>.

In cubic spline fit we use a cubic polynomial $f_i(x)$ to represent the function in each interval (x_i, x_{i+1}). Thus if there are n points at which a function has values tabulated points being

$x_1, x_2, \ldots, x_n$ there are $(n-1)$ intervals and $(n-1)$ cubic splines $f_1(x), f_2(x), \cdots f_{n-1}(x)$ where $f_1(x)$ is the cubic for the interval (x_1, x_2) and $f_{n-1}(x)$ is the cubic for the interval (x_{n-1}, x_n).

The cubic polynomial chosen is such that in each interior point $x_{i+1}, x_{i+2}, \ldots x_{n-1}$ called knots the following conditions are satisfied.

(i) $f_i(x_{i+1}) = f_{i+1}(x_{i+1})$, where $f_i(x)$ is the polynomial for the interval (x_i, x_{i+1}) and $f_{i+1}(x)$ is the polynomial for the interval (x_{i+1}, x_{i+2}), $(i = 1 \text{ to } n-2)$ —— ①

(ii) $f_i'(x_{i+1}) = f_{i+1}'(x_{i+1})$, $(i = 1 \text{ to } n-2)$ —— ②

(iii) $f_i''(x_{i+1}) = f_{i+1}''(x_{i+1})$, $(i = 1 \text{ to } n-2)$ —— ③

The equations for the polynomials in each interval are derived by first observing that the second derivative of $f_i(x)$ are straight lines in each interval.

The equations for the second derivative may be represented by

$$f_i''(x) = a_i \frac{x - x_{i+1}}{x_i - x_{i+1}} + a_{i+1} \frac{x - x_i}{x_{i+1} - x_i}, \quad i = 1 \text{ to } n-1$$

—— ④

where, $a_i = f''(x_i)$ and

$a_{i+1} = f''(x_{i+1})$

Integrating eqn (4) twice, we get

$$f_i(x) = \frac{f''(x_i)\,(x-x_{i+1})^3}{6(x_i-x_{i+1})} + \frac{f''(x_{i+1})\,(x-x_i)^3}{6(x_{i+1}-x_i)}$$

$$+ A(x-x_i) + B(x-x_{i+1}) \quad \text{——— (5)}$$

where A and B are two constants of integration to be determined by putting the values of $f_i(x)$ at $x = x_i$ and $x = x_{i+1}$, given in tables.

$$f_i(x_i) = \frac{f''(x_i)\,(x_i-x_{i+1})^3}{6(x_i-x_{i+1})} + 0 + 0 + B(x_i-x_{i+1})$$

$$\Rightarrow B = \frac{f(x_i)}{(x_i-x_{i+1})} - \frac{f''(x_i)}{6}(x_i-x_{i+1})$$

and

$$f(x_{i+1}) = \frac{f''(x_{i+1})\,(x_{i+1}-x_i)^3}{6(x_{i+1}-x_i)} + A(x_{i+1}-x_i)$$

$$\Rightarrow \quad A = \frac{f(x_{i+1})}{(x_{i+1}-x_i)} - f''(x_{i+1})(x_{i+1}-x_i)$$

$\therefore$ eqn (5) $\Rightarrow$

$$f_i(x) = \frac{f''(x_i)\,(x-x_{i+1})^3}{6(x_i-x_{i+1})} + \frac{f''(x_{i+1})\,(x-x_i)^3}{6(x_{i+1}-x_i)}$$

$$+ \left[\frac{f(x_{i+1})}{(x_{i+1}-x_i)} - \frac{f''(x_{i+1})}{6}(x_{i+1}-x_i)\right](x-x_i)$$

$$+ \left[\frac{f(x_i)}{(x_i-x_{i+1})} - \frac{f''(x_i)}{6}(x_i-x_{i+1})\right](x-x_{i+1})$$

$$(i = 1, 2, \ldots, n-1) \quad \text{——— (6)}$$

Equation (6) is the cubic eqn for interval (x_i, x_{i+1}). As there are n points $x_1, x_2, \ldots, x_n$ there are total $(n-1)$ such cubic eqn's to be determined to fit a curve passing through all the n points.

The cubic equations are not fully determined because, $f''(x_1), f''(x_2), \ldots, f''(x_n)$ are not known. These can be obtained by computing the condition given in eqn ②. By differentiating eqn ⑤ and setting $f_i'(x)$ and $f_{i+1}'(x)$ equal at each knot $x_2, x_3, x_3, \ldots x_{n-1}$.

$$(x_{i+1} - x_i) f''(x_i) + 2(x_{i+2} - x_i) f''(x_{i+1}) + (x_{i+2} - x_{i+1}) f''(x_{i+2}) =$$

$$\frac{6}{(x_{i+2} - x_{i+1})}[f(x_{i+2}) - f(x_{i+1})] - \frac{6}{(x_{i+1} - x_i)}[f(x_{i+1}) - f(x_i)] \quad (7)$$

for $(i = 1 \text{ to } n-2)$

These give $(n-2)$ eqns for n unknown $f''(x_i)$s. The solution will thus not be unique. The soln will become unique, if we assume $f''(x_1)$ and $f''(x_n)$ are not zero, which means that we are assuming that the curve starts as a straight line at x_1 and ends as a straight line at x_n.

Such a spline is known as a natural spline.

With this assumption, we get $(n-2)$ eqns
~~ooo~~ in $(n-2)$ unknowns and thus a unique solution.
The simultaneous equations (7) have a tridiagonal form making them easy to solve.

After solving for $f''(x)$, we put these values in eqn (6) and obtain the equations for the splines.